

TILTING THEORY AND FUNCTOR CATEGORIES III. THE MAPS CATEGORY.

R. MARTÍNEZ-VILLA AND M. ORTIZ-MORALES

ABSTRACT. In this paper we continue the project of generalizing tilting theory to the category of contravariant functors $\text{Mod}(\mathcal{C})$, from a skeletally small preadditive category \mathcal{C} to the category of abelian groups, initiated in [17]. In [18] we introduced the notion of a generalized tilting category \mathcal{T} , and extended Happel's theorem to $\text{Mod}(\mathcal{C})$. We proved that there is an equivalence of triangulated categories $D^b(\text{Mod}(\mathcal{C})) \cong D^b(\text{Mod}(\mathcal{T}))$. In the case of dualizing varieties, we proved a version of Happel's theorem for the categories of finitely presented functors. We also proved in this paper, that there exists a relation between covariantly finite coresolving categories, and generalized tilting categories. Extending theorems for artin algebras proved in [4], [5]. In this article we consider the category of maps, and relate tilting categories in the category of functors, with relative tilting in the category of maps. Of special interest is the category $\text{mod}(\text{mod } \Lambda)$ with Λ an artin algebra.

1. INTRODUCTION AND BASIC RESULTS

This is the last article in a series of three in which, having in mind applications to the category of functors from subcategories of modules over a finite dimensional algebra to the category of abelian groups, we generalize tilting theory, from rings to functor categories.

In the first paper [17] we generalized classical tilting to the category of contravariant functors from a preadditive skeletally small category \mathcal{C} , to the category of abelian groups and generalized Bongartz's proof [10] of Brenner-Butler's theorem [11]. We then applied the theory so far developed, to the study of locally finite infinite quivers with no relations, and computed the Auslander-Reiten components of infinite Dynkin diagrams. Finally, we applied our results to calculate the Auslander-Reiten components of the category of Koszul functors (see [19], [20], [21]) on a regular component of a finite dimensional algebra over a field. These results generalize the theorems on the preprojective algebra obtained in [15].

Following [12], in [18] we generalized the proof of Happel's theorem given by Cline, Parshall and Scott: given a generalized tilting subcategory \mathcal{T} of $\text{Mod}(\mathcal{C})$, the derived categories of bounded complexes $D^b(\text{Mod}(\mathcal{C}))$ and $D^b(\text{Mod}(\mathcal{T}))$ are equivalent, and we discussed a partial converse [14]. We also saw that for a dualizing

Date: January 21, 2010.

1991 *Mathematics Subject Classification.* 2000]Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Classical Tilting, Functor Categories.

The second author thanks CONACYT for giving him financial support during his graduate studies .

This paper is in final form and no version of it will be submitted for publication elsewhere.

variety \mathcal{C} and a tilting subcategory $\mathcal{T} \subset \text{mod}(\mathcal{C})$ with pseudokernels, the categories of finitely presented functors $\text{mod}(\mathcal{C})$ and $\text{mod}(\mathcal{T})$ have equivalent derived bounded categories, $D^b(\text{mod}(\mathcal{C})) \cong D^b(\text{mod}(\mathcal{T}))$. Following closely the results for artin algebras obtained in [3], [4], [5], by Auslander, Buchweits and Reiten, we end the paper proving that for a Krull-Schmidt dualizing variety \mathcal{C} , there are analogous relations between covariantly finite subcategories and generalized tilting subcategories of $\text{mod}(\mathcal{C})$.

This paper is dedicated to study tilting subcategories of $\text{mod}(\mathcal{C})$. In order to have a better understanding of these categories, we use the relation between the categories $\text{mod}(\mathcal{C})$ and the category of maps, $\text{maps}(\mathcal{C})$, given by Auslander in [1]. Of special interest is the case when \mathcal{C} is the category of finitely generated left Λ -modules over an artin algebra Λ , since in this case the category $\text{maps}(\mathcal{C})$ is equivalent to the category of finitely generated Γ -modules, $\text{mod}(\Gamma)$, over the artin algebra of triangular matrices $\Gamma = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$. In this situation, tilting subcategories on $\text{mod}(\text{mod}(\Lambda))$ will correspond to relative tilting subcategories of $\text{mod}(\Gamma)$, which in principle, are easier to compute.

The paper consists of three sections:

In the first section we establish the notation and recall some basic concepts. In the second one, for a variety of annuli with pseudokernels \mathcal{C} , we prove that generalized tilting subcategories of $\text{mod}(\mathcal{C})$ are in correspondence with relative tilting subcategories of $\text{maps}(\mathcal{C})$ [9]. In the third section, we explore the connections between $\text{mod } \Gamma$, with $\Gamma = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$ and the category $\text{mod}(\text{mod}(\Lambda))$. We compare the Auslander-Reiten sequences in $\text{mod}(\Gamma)$ with Auslander-Reiten sequences in $\text{mod}(\text{mod}(\Lambda))$. We end the paper proving that, some important subcategories of $\text{mod}(\mathcal{C})$ related with tilting, like: contravariantly, covariantly, functorially finite [see 18], correspond to subcategories of $\text{maps}(\mathcal{C})$ with similar properties.

1.1. Functor Categories. In this subsection we will denote by \mathcal{C} an arbitrary skeletally small pre additive category, and $\text{Mod}(\mathcal{C})$ will be the category of contravariant functors from \mathcal{C} to the category of abelian groups. The subcategory of $\text{Mod}(\mathcal{C})$ consisting of all finitely generated projective objects, $\mathfrak{p}(\mathcal{C})$, is a skeletally small additive category in which idempotents split, the functor $P : \mathcal{C} \rightarrow \mathfrak{p}(\mathcal{C})$, $P(C) = \mathcal{C}(-, C)$, is fully faithful and induces by restriction $\text{res} : \text{Mod}(\mathfrak{p}(\mathcal{C})) \rightarrow \text{Mod}(\mathcal{C})$, an equivalence of categories. For this reason, we may assume that our categories are skeletally small, additive categories, such that idempotents split. Such categories were called **annuli varieties** in [2], for short, varieties.

To fix the notation, we recall known results on functors and categories that we use through the paper, referring for the proofs to the papers by Auslander and Reiten [1], [4], [5].

Given a category \mathcal{C} we will write for short, $\mathcal{C}(-, ?)$ instead of $\text{Hom}_{\mathcal{C}}(-, ?)$ and when it is clear from the context we use just $(-, ?)$.

Definition 1.1. *Given a variety \mathcal{C} , we say \mathcal{C} has **pseudokernels**; if given a map $f : C_1 \rightarrow C_0$, there exists a map $g : C_2 \rightarrow C_1$ such that the sequence of representable functors $\mathcal{C}(-, C_2) \xrightarrow{(-, g)} \mathcal{C}(-, C_1) \xrightarrow{(-, f)} \mathcal{C}(-, C_0)$ is exact.*

A functor M is **finitely presented**; if there exists an exact sequence

$$\mathcal{C}(-, C_1) \rightarrow \mathcal{C}(-, C_0) \rightarrow M \rightarrow 0$$

We denote by $\text{mod}(\mathcal{C})$ the full subcategory of $\text{Mod}(\mathcal{C})$ consisting of finitely presented functors. It was proved in [1] $\text{mod}(\mathcal{C})$ is abelian, if and only if, \mathcal{C} has pseudokernels.

1.2. Krull-Schmidt Categories. We start giving some definitions from [6].

Definition 1.2. Let R be a commutative artin ring. An R -variety \mathcal{C} , is a variety such that $\mathcal{C}(C_1, C_2)$ is an R -module, and composition is R -bilinear. Under these conditions $\text{Mod}(\mathcal{C})$ is an R -variety, which we identify with the category of contravariant functors $(\mathcal{C}^{op}, \text{Mod}(R))$.

An R -variety \mathcal{C} is **Hom-finite**, if for each pair of objects C_1, C_2 in \mathcal{C} , the R -module $\mathcal{C}(C_1, C_2)$ is finitely generated. We denote by $(\mathcal{C}^{op}, \text{mod}(R))$, the full subcategory of $(\mathcal{C}^{op}, \text{Mod}(R))$ consisting of the \mathcal{C} -modules such that; for every C in \mathcal{C} the R -module $M(C)$ is finitely generated. The category $(\mathcal{C}^{op}, \text{mod}(R))$ is abelian and the inclusion $(\mathcal{C}^{op}, \text{mod}(R)) \rightarrow (\mathcal{C}^{op}, \text{Mod}(R))$ is exact.

The category $\text{mod}(\mathcal{C})$ is a full subcategory of $(\mathcal{C}^{op}, \text{mod}(R))$. The functors $D : (\mathcal{C}^{op}, \text{mod}(R)) \rightarrow (\mathcal{C}, \text{mod}(R))$, and $D : (\mathcal{C}, \text{mod}(R)) \rightarrow (\mathcal{C}^{op}, \text{mod}(R))$, are defined as follows: for any C in \mathcal{C} , $D(M)(C) = \text{Hom}_R(M(C), I(R/r))$, with r the Jacobson radical of R , and $I(R/r)$ is the injective envelope of R/r . The functor D defines a duality between $(\mathcal{C}, \text{mod}(R))$ and $(\mathcal{C}^{op}, \text{mod}(R))$. If \mathcal{C} is an Hom-finite R -category and M is in $\text{mod}(\mathcal{C})$, then $M(C)$ is a finitely generated R -module and it is therefore in $\text{mod}(R)$.

Definition 1.3. An Hom-finite R -variety \mathcal{C} is **dualizing**, if the functor

$$D : (\mathcal{C}^{op}, \text{mod}(R)) \rightarrow (\mathcal{C}, \text{mod}(R))$$

induces a duality between the categories $\text{mod}(\mathcal{C})$ and $\text{mod}(\mathcal{C}^{op})$.

It is clear from the definition that for dualizing categories \mathcal{C} the category $\text{mod}(\mathcal{C})$ has enough injectives.

To finish, we recall the following definition:

Definition 1.4. An additive category \mathcal{C} is **Krull-Schmidt**, if every object in \mathcal{C} decomposes in a finite sum of objects whose endomorphism ring is local.

In [18 Theo. 2] we see that for a dualizing Krull-Schmidt variety the finitely presented functors have projective covers.

Theorem 1.5. Let \mathcal{C} a dualizing Krull-Schmidt R -variety. Then $\text{mod}(\mathcal{C})$ is a dualizing Krull-Schmidt variety.

1.3. Contravariantly finite categories. [4] Let \mathcal{X} be a subcategory of $\text{mod}(\mathcal{C})$, which is closed under summands and isomorphisms. A morphism $f : X \rightarrow M$ in $\text{mod}(\mathcal{C})$, with X in \mathcal{X} , is a *right \mathcal{X} -approximation* of M , if $(-, X)_{\mathcal{X}} \xrightarrow{(-, h)_{\mathcal{X}}} (-, M)_{\mathcal{X}} \rightarrow 0$ is an exact sequence, where $(-, ?)_{\mathcal{X}}$ denotes the restriction of $(-, ?)$ to the category \mathcal{X} . Dually, a morphism $g : M \rightarrow X$, with X in \mathcal{X} , is a *left \mathcal{X} -approximation* of M , if $(X, -)_{\mathcal{X}} \xrightarrow{(g, -)_{\mathcal{X}}} (M, -)_{\mathcal{X}} \rightarrow 0$ is exact.

A subcategory \mathcal{X} of $\text{mod}(\mathcal{C})$ is called *contravariantly (covariantly) finite* in $\text{mod}(\mathcal{C})$, if every object M in $\text{mod}(\mathcal{C})$ has a right (left) \mathcal{X} -approximation; and *functorially finite*, if it is both contravariantly and covariantly finite.

A subcategory \mathcal{X} of $\text{mod}(\mathcal{C})$ is *resolving (coresolving)*, if it satisfies the following three conditions: (a) it is closed under extensions, (b) it is closed under kernels of

epimorphisms (cokernels of monomorphisms), and (c) it contains the projective (injective) objects.

1.4. Relative Homological Algebra and Frobenius Categories. In this subsection we recall some results on relative homological algebra introduced by Auslander and Solberg in [9],[see also 14, 23].

Let \mathcal{C} be an additive category which is embedded as a full subcategory of an abelian category \mathcal{A} , and suppose that \mathcal{C} is closed under extensions in \mathcal{A} . Let \mathcal{S} be a collection of exact sequences in \mathcal{A}

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

f is called an *admissible monomorphism*, and g is called an *admissible epimorphism*. A pair $(\mathcal{C}, \mathcal{S})$ is called an *exact category* provided that: (a) Any split exact sequence whose terms are in \mathcal{C} is in \mathcal{S} . (b) The composition of admissible monomorphisms (resp., epimorphisms) is an admissible monomorphism (resp., epimorphism). (c) It is closed under pullbacks (pushouts) of admissible epimorphisms (admissible monomorphisms).

Let $(\mathcal{C}, \mathcal{S})$ be an exact subcategory of an abelian category \mathcal{A} . Since the collection \mathcal{S} is closed under pushouts, pullbacks and Baer sums, it gives rise to a subfunctor F of the additive bifunctor $\text{Ext}_{\mathcal{C}}^1(-, -) : \mathcal{C} \times \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ [9]. Given such a functor F , we say that an exact sequence $\eta : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{C} is F -exact, if η is in $F(\mathcal{C}, \mathcal{A})$, we will write some times $\text{Ext}_F^1(-, ?)$ instead of $F(-, ?)$. An object P in \mathcal{C} is F -projective, if for each F -exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the sequence $0 \rightarrow (P, N) \rightarrow (P, E) \rightarrow (P, M) \rightarrow 0$ is exact. Analogously we have the definition of an F -injective object.

If for any object C in \mathcal{C} there is an F -exact sequence $0 \rightarrow A \rightarrow P \rightarrow C \rightarrow 0$, with P an F -projective, then we say $(\mathcal{C}, \mathcal{S})$ has enough F -projectives. Dually, if for any object C in \mathcal{C} there is an F -exact sequence $0 \rightarrow C \rightarrow I \rightarrow A \rightarrow 0$, with I an F -injective, then $(\mathcal{C}, \mathcal{S})$ has enough F -injectives.

An exact category $(\mathcal{C}, \mathcal{S})$ is called *Frobenius*, if the category $(\mathcal{C}, \mathcal{S})$ has enough F -projectives and enough F -injectives and they coincide.

Let F be a subfunctor of $\text{Ext}_{\mathcal{C}}^1(-, -)$. Suppose F has enough projectives. Then for any C in \mathcal{C} there is an exact sequence in \mathcal{C} of the form

$$\cdots P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} C \rightarrow 0$$

where P_i is F -projective for $i \geq 0$ and $0 \rightarrow \text{Im} d_{i+1} \rightarrow P_i \rightarrow \text{Im} d_i \rightarrow 0$ is F -exact for all $i \geq 0$. Such sequence is called an *F -exact projective resolution*. Analogously we have the definition of an *F -exact injective resolution*.

When $(\mathcal{C}, \mathcal{S})$ has enough F -injectives (enough F -projectives), using F -exact injective resolutions (respectively, F -exact projective resolutions), we can prove that for any object C in \mathcal{C} , (A in \mathcal{C}), there exists a right derived functor of $\text{Hom}_{\mathcal{C}}(C, -)$ ($\text{Hom}_{\mathcal{C}}(-, A)$).

We denote by $\text{Ext}_F^i(C, -)$ the right derived functors of $\text{Hom}_{\mathcal{C}}(C, -)$ and by $\text{Ext}_F^i(-, A)$ the right derived functors of $\text{Hom}_{\mathcal{C}}(-, A)$.

2. THE MAPS CATEGORY, $\text{maps}(\mathcal{C})$

In this section \mathcal{C} is an annuli variety with pseudokernels. We will study tilting subcategories of $\text{mod}(\mathcal{C})$ via the equivalence of categories between the maps category, module the homotopy relation, and the category of functors, $\text{mod}(\mathcal{C})$, given

by Auslander in [1]. We will provide $\text{maps}(\mathcal{C})$ with a structure of exact category such that, tilting subcategories of $\text{mod}(\mathcal{C})$ will correspond to relative tilting subcategories of $\text{maps}(\mathcal{C})$. We begin the section recalling concepts and results from [1], [14] and [23].

The objects in $\text{maps}(\mathcal{C})$ are morphisms $(f_1, A_1, A_0) : A_1 \xrightarrow{f_1} A_0$, and the maps are pairs $(h_1, h_0) : (f_1, A_1, A_0) \rightarrow (g_1, B_1, B_0)$, such that the following square commutes

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_0 \\ h_1 \downarrow & & h_0 \downarrow \\ B_1 & \xrightarrow{g_1} & B_0 \end{array}$$

We say that two maps $(h_1, h_0), (h'_1, h'_0) : (f_1, A_1, A_0) \rightarrow (g_1, B_1, B_0)$ are homotopic, if there exist a morphisms $s : A_0 \rightarrow B_1$ such that $h_0 - h'_0 = g_1 s$. Denote by $\underline{\text{maps}}(\mathcal{C})$ the category of maps modulo the homotopy relation. It was proved in [1] that the categories $\underline{\text{maps}}(\mathcal{C})$ and $\text{mod}(\mathcal{C})$ are equivalent. The equivalence is given by a functor $\underline{\Phi} : \underline{\text{maps}}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$ induced by the functor $\Phi : \text{map}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$ given by

$$\Phi(A_1 \xrightarrow{f_1} A_0) = \text{Coker}((-, A_1) \xrightarrow{(-, f_1)} (-, A_0)).$$

The category $\text{maps}(\mathcal{C})$ is not in general an exact category, we will use instead the exact category $P^0(\mathcal{A})$ of projective resolutions, which module the homotopy relation, is equivalent to $\underline{\text{maps}}(\mathcal{C})$.

Since we are assuming \mathcal{C} has pseudokernels, the category $\mathcal{A} = \text{mod}(\mathcal{C})$ is abelian. We can consider the categories of complexes $C(\mathcal{A})$, and its subcategory $C^-(\mathcal{A})$, of bounded above complexes, both are abelian. Moreover, if we consider the class of exact sequences $\mathcal{S} : 0 \rightarrow L \xrightarrow{j} M \xrightarrow{\pi} N \rightarrow 0$, such that, for every k , the exact sequences $0 \rightarrow L_k \xrightarrow{j_k} M_k \xrightarrow{\pi_k} N_k \rightarrow 0$ split, then $(\mathcal{S}, C(\mathcal{A}))$, $(\mathcal{S}, C^-(\mathcal{A}))$ are exact categories with enough projectives, in fact they are both Frobenius. In the first case the projective are summands of complexes of the form:

$$\cdots B_{k+2} \coprod B_{k+1} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} B_{k+1} \coprod B_k \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} B_k \coprod B_{k-1} \cdots$$

In the second case of the form:

$$\cdots B_{k+3} \coprod B_{k+2} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} B_{k+2} \coprod B_{k+1} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} B_{k+1} \coprod B_k \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} B_k \rightarrow 0$$

If we denote by $\underline{C}^-(\mathcal{A})$ the stable category, it is well known [23], [14], that the homotopy category $K^-(\mathcal{A})$ and $\underline{C}^-(\mathcal{A})$ are equivalent.

Now, denote by $P^0(\mathcal{A})$ the full subcategory of $C^-(\mathcal{A})$ consisting of projective resolutions, this is, complexes of projectives P :

$$\cdots P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

such that $H^i(P) = 0$ for $i \neq 0$. Then we have the following:

Proposition 2.1. *The category $P^0(\mathcal{A})$ is closed under extensions and kernels of epimorphisms.*

Proof. If $0 \rightarrow P. \rightarrow E. \rightarrow Q. \rightarrow 0$ is an exact sequence in $P^0(\mathcal{A})$, then $0 \rightarrow P_j \rightarrow E_j \rightarrow Q_j \rightarrow 0$ is a splitting exact sequence in \mathcal{A} with P_j, Q_j projectives, hence E_j is also projective. By the long homology sequence we have the exact sequence: $\cdots \rightarrow H^i(P.) \rightarrow H^i(E.) \rightarrow H^i(Q.) \rightarrow H^{i-1}(P.) \rightarrow \cdots$, with $H^i(P.) = H^i(Q.) = 0$, for $i \neq 0$. This implies $E. \in P^0(\mathcal{A})$.

Now, let $0 \rightarrow T. \rightarrow Q. \rightarrow P. \rightarrow 0$ be an exact sequence with $Q., P.$ in $P^0(\mathcal{A})$. This implies that for each k , $0 \rightarrow T_k \rightarrow Q_k \rightarrow P_k \rightarrow 0$ is an exact and splittable sequence, hence each T_k is projective and, by the long homology sequence, we have the following exact sequence

$$\cdots \rightarrow H^1(T.) \rightarrow H^1(Q.) \rightarrow H^1(P.) \rightarrow H^0(T.) \rightarrow H^0(Q.) \rightarrow H^0(P.) \rightarrow 0$$

with $H^{i+1}(P.) = H^i(Q.) = 0$ for $i \geq 1$. This implies $H^i(T.) = 0$, for $i \neq 0$. \square

If $\mathcal{S}_{P^0(\mathcal{A})}$ denotes the collection of exact sequences with objects in $P^0(\mathcal{A})$, then $(P^0(\mathcal{A}), \mathcal{S}_{P^0(\mathcal{A})})$ is an exact subcategory of $(C^-(\mathcal{A}), \mathcal{S})$. The category $P^0(\mathcal{A})$ has enough projectives, they are the complexes of the form:

$$(2.1) \quad \cdots \rightarrow P_3 \amalg P_2 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} P_2 \amalg P_1 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} P_1 \amalg P_0 \rightarrow 0$$

Denote by $R^0(\mathcal{A})$ the category $P^0(\mathcal{A})$ module the homotopy relation. This is: $R^0(\mathcal{A})$ is a full subcategory of $\underline{\mathcal{C}}^-(\mathcal{A}) = K^-(\mathcal{A})$. It is easy to check that $R^0(\mathcal{A})$ is the category with objects in $P^0(\mathcal{A})$ and maps the maps of complexes, module the maps that factor through a complex of the form:

$$\cdots \rightarrow P_3 \amalg P_2 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} P_2 \amalg P_1 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} P_1 \amalg P_0 \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} P_0 \rightarrow 0$$

We have the following:

Proposition 2.2. *There is a functor $\Psi : P^0(\mathcal{A}) \rightarrow \text{maps}(\mathcal{C})$ which induces an equivalence of categories $\underline{\Psi} : R^0(\mathcal{A}) \rightarrow \underline{\text{maps}}(\mathcal{C})$ given by:*

$$\Psi(P.) = \Psi(\cdots \rightarrow (-, A_2) \xrightarrow{(-, f_2)} (-, A_1) \xrightarrow{(-, f_1)} (-, A_0) \rightarrow 0) = A_1 \xrightarrow{f_1} A_0$$

Proof. Since \mathcal{C} has pseudokernels, any map $A_1 \xrightarrow{f_1} A_0$ induces an exact sequence

$$(-, A_n) \xrightarrow{(-, f_n)} (-, A_{n-1}) \rightarrow \cdots \rightarrow (-, A_2) \xrightarrow{(-, f_2)} (-, A_1) \xrightarrow{(-, f_1)} (-, A_0)$$

and Ψ is clearly dense. Let $(-, \varphi) : P. \rightarrow Q.$ be a map of complexes in $P^0(\mathcal{A})$:

$$(2.2) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & (-, A_2) & \xrightarrow{(-, f_2)} & (-, A_1) & \xrightarrow{(-, f_1)} & (-, A_0) \longrightarrow 0 \\ & & \downarrow (-, \varphi_2) & & \downarrow (-, \varphi_1) & & \downarrow (-, \varphi_0) \\ \cdots & \longrightarrow & (-, B_2) & \xrightarrow{(-, g_2)} & (-, B_1) & \xrightarrow{(-, g_1)} & (-, B_0) \longrightarrow 0 \end{array}$$

If $\Psi(P. \xrightarrow{(-, \varphi)} Q.)$ is homotopic to zero, then we have a map $s_0 : A_0 \rightarrow A_1$ such that $g_0 s_0 = \varphi_0$:

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_0 \\ \varphi_1 \downarrow & \swarrow s_0 & \searrow \varphi_0 \downarrow \\ B_1 & \xrightarrow{g_1} & B_0 \end{array}$$

and s_0 lifts to a homotopy $s : P. \rightarrow Q.$. Conversely, any homotopy $s : P. \rightarrow Q.$ induces an homotopy in $\text{maps}(\mathcal{C})$. Then Ψ is faithful.

If $\Psi(P.) = (f_1, A_1, A_0)$, $\Psi(Q.) = (g_1, B_1, B_0)$ and $(h_0, h_1) : \Psi(P.) \rightarrow \Psi(Q.)$ is a map in $\text{maps}(\mathcal{C})$, then (h_0, h_1) lifts to a map $(-, h) = (-, h_i) : P. \rightarrow Q.$, and Ψ is full. \square

Corollary 2.3. *There is an equivalence of categories $\Theta : R^0(\mathcal{A}) \rightarrow \text{mod}(\mathcal{C})$ given by $\underline{\Theta} = \underline{\Phi\Psi}$, with $\Theta = \Phi\Psi$.*

Proposition 2.4. *Let $P.$ be an object in $P^0(\mathcal{A})$, denote by $\text{rpdim}P.$ the relative projective dimension of $P.$, and by $\text{pdim}\Theta(P.)$ the projective dimension of $\Theta(P.)$. Then we have $\text{rpdim}P. = \text{pdim}\Theta(P.)$. Moreover, if $\Omega^i(P.)$ is the relative syzygy of $P.$, then for all $i \geq 0$, we have $\Omega^i(\Theta(P.)) = \Theta(\Omega^i(P.))$.*

Proof. Let $P.$ be the complex resolution

$$0 \rightarrow (-, A_n) \xrightarrow{(-, f_n)} (-, A_{n-1}) \rightarrow \cdots \rightarrow (-, A_2) \xrightarrow{(-, f_2)} (-, A_1) \xrightarrow{(-, f_1)} (-, A_0) \rightarrow 0$$

and $M = \text{Coker}(-, f_1)$, then $\text{pdim}M \leq n$.

Now, consider the following commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & \longrightarrow & A_n \\ & & & & \parallel & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & & & & A_n & \xrightarrow{-\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & A_n \coprod A_{n-1} \\ & & & & \vdots & & \vdots \\ & & & & A_n & \xrightarrow{\cdots} & A_3 \xrightarrow{f_3} A_2 \\ & & & & \downarrow \begin{pmatrix} 1 \\ f_n \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ f_3 \end{pmatrix} \\ & & & & A_n & \xrightarrow{-\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & A_n \coprod A_{n-1} \xrightarrow{\cdots} A_3 \coprod A_2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} A_2 \coprod A_1 \\ & & & & \parallel & & \downarrow \begin{pmatrix} 1 \\ f_n \end{pmatrix} \\ & & & & A_n & \xrightarrow{f_n} & A_{n-1} \xrightarrow{\cdots} A_2 \xrightarrow{f_2} A_1 \\ & & & & \downarrow \begin{pmatrix} 1 \\ f_n \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ f_2 \end{pmatrix} \\ & & & & A_n & \xrightarrow{-\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & A_n \coprod A_{n-1} \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} A_{n-1} \coprod A_{n-2} \xrightarrow{\cdots} A_2 \coprod A_1 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} A_1 \coprod A_0 \\ & & & & \parallel & & \downarrow \begin{pmatrix} 1 \\ f_n \end{pmatrix} \\ & & & & A_n & \xrightarrow{f_n} & A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{\cdots} A_1 \xrightarrow{f_1} A_0 \end{array}$$

Set $\mathbf{Q}_n = 0 \rightarrow (-, A_n) \rightarrow 0$, and for $n-1 \geq i \geq 1$ consider the following complex \mathbf{Q}_i :

$$\begin{aligned} & 0 \rightarrow (-, A_n) \rightarrow (-, A_n) \coprod (-, A_{n-1}) \rightarrow \cdots \rightarrow \\ & \rightarrow (-, A_{i+2}) \coprod (-, A_{i+1}) \rightarrow (-, A_{i+1}) \coprod (-, A_i) \rightarrow 0 \end{aligned}$$

Then we have a relative projective resolution

$$0 \rightarrow \mathbf{Q}_n \rightarrow \mathbf{Q}_{n-1} \rightarrow \cdots \rightarrow \mathbf{Q}_1 \rightarrow \mathbf{Q}_0 \rightarrow P. \rightarrow 0$$

with relative syzygy the complex:

$$\Omega^i(P.) : 0 \rightarrow (-, A_n) \rightarrow (-, A_{n-1}) \rightarrow (-, A_{n-2}) \cdots (-, A_{i+2}) \rightarrow (-, A_{i+1}) \rightarrow 0$$

for $n-1 \geq i \geq 0$.

Therefore: we have an exact sequence

$$0 \rightarrow \Theta(\Omega(P.)) \rightarrow \Theta(\mathbf{Q}_0) \rightarrow \Theta(P.) \rightarrow 0$$

in $\text{mod}(\mathcal{C})$. Since $\Theta(\mathbf{Q}_i) = (-, A_i)$ and $\Theta(P.) = M$, we have $\Omega(\Theta(P.)) = \Theta(\Omega(P.))$, and we can prove by induction that $\Omega^i(\Theta(P.)) = \Theta(\Omega^i P.)$, for all $i \geq 0$. It follows $\text{rpdim} P. \geq \text{pdim} \Theta(P.)$.

Conversely, applying Θ to a relative projective resolution

$$0 \rightarrow \mathbf{Q}_n \rightarrow \mathbf{Q}_{n-1} \rightarrow \cdots \rightarrow \mathbf{Q}_1 \rightarrow \mathbf{Q}_0 \rightarrow P. \rightarrow 0,$$

we obtain a projective resolution of $\Theta(P.)$

$$0 \rightarrow \Theta(\mathbf{Q}_n) \rightarrow \Theta(\mathbf{Q}_{n-1}) \rightarrow \cdots \rightarrow \Theta(\mathbf{Q}_1) \rightarrow \Theta(\mathbf{Q}_0) \rightarrow \Theta(P.) \rightarrow 0.$$

It follows $\text{rpdim} P. \leq \text{pdim} \Theta(P.)$. \square

As a corollary we have:

Corollary 2.5. *Let \mathcal{C} a dualizing Krull-Schmidt variety. If $P.$ and $Q.$ are complexes in $P^0(\text{mod}(\mathcal{C}))$ without summands of the form (2.1), then there is an isomorphism*

$$\text{Ext}_{\mathcal{C}-(\text{mod}(\mathcal{C}))}^k(P., Q.) = \text{Ext}_{\text{mod}(\mathcal{C})}^k(\Theta(P.), \Theta(Q.))$$

Proof. By Proposition 2.4, we see that $\Theta(\Omega^i P.) = \Omega^i(\Theta(P.))$, $i \geq 0$. It is enough to prove the corollary for $k = 1$. Assume that $(*)$ $0 \rightarrow Q. \xrightarrow{(-, j_i)} E. \xrightarrow{(-, p_i)} P. \rightarrow 0$ is a exact sequence in $\text{Ext}_{\mathcal{C}-(\text{mod}(\mathcal{C}))}^k(P., Q.)$, with $Q. = \cdots \rightarrow (-, B_2) \xrightarrow{(-, g_2)} (-, B_1) \xrightarrow{(-, g_1)} (-, B_0) \rightarrow 0$, $P. = \cdots \rightarrow (-, A_2) \xrightarrow{(-, f_2)} (-, A_1) \xrightarrow{(-, f_1)} (-, A_0) \rightarrow 0$, $E. = \cdots \rightarrow (-, E_2) \xrightarrow{(-, h_2)} (-, E_1) \xrightarrow{(-, h_1)} (-, E_0) \rightarrow 0$. Since the exact sequence $0 \rightarrow (-, B_i) \xrightarrow{(-, j_i)} (-, E_i) \xrightarrow{(-, p_i)} (-, A_i) \rightarrow 0$ splits $E_i = A_i \amalg B_i$, $i \geq 0$. Then we have an exact sequence in $\text{mod}(\mathcal{C})$

$$(2.3) \quad 0 \rightarrow \Theta(Q.) \xrightarrow{\rho} \Theta(E.) \xrightarrow{\sigma} \Theta(P.) \rightarrow 0$$

If (2.3) splits, then there exist a map $\delta : \Theta(E.) \rightarrow \Theta(Q.)$ such that $\delta\rho = 1_{\Theta(Q.)}$. We have a lifting of δ , $(-, l_i)_{i \in \mathbb{Z}} : E. \rightarrow Q.$ such that the following diagram is commutative

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (-, B_1) & \xrightarrow{(-, g_1)} & (-, B_0) & \xrightarrow{\pi} & \Theta(Q.) \longrightarrow 0 \\ & & \downarrow (-, l_1 j_i) & & \downarrow (-, l_0 j_0) & & \parallel \\ \cdots & \longrightarrow & (-, B_1) & \xrightarrow{(-, g_1)} & (-, B_0) & \xrightarrow{\pi} & \Theta(Q.) \longrightarrow 0 \end{array}$$

The complex $Q.$ has not summand of the form (2.1), hence, $Q.$ is a minimal projective resolution of $\Theta(Q.)$.

Since $\pi : (-, B_0) \rightarrow \Theta(Q.)$ is a projective cover, the map $(-, l_0 j_0) : (-, B_0) \rightarrow (-, B_0)$ is an isomorphism, and it follows by induction that all maps $(-, l_i j_i)$ are isomorphisms, which implies that the map $\{(-, j_i)\}_{i \in \mathbb{Z}} : Q. \rightarrow E.$ is a splitting homomorphism of complexes.

Given an exact sequence $(**) 0 \rightarrow G \rightarrow H \rightarrow F \rightarrow 0$, in $\text{mod}(\mathcal{C})$, we take minimal projective resolutions $P.$ and $Q.$ of F and G , respectively, by the Horseshoe's Lemma, we have a projective resolution $E.$ for H , with $E_i = Q_i \oplus P_i$, and $0 \rightarrow \Theta(Q.) \rightarrow \Theta(E.) \rightarrow \Theta(P.) \rightarrow 0$ is a exact sequence in $\text{mod}(\mathcal{C})$ isomorphic to $(**)$. \square

2.1. Relative Tilting in $\text{maps}(\mathcal{C})$. Let \mathcal{C} a dualizing Krull-Schmidt variety. In order to define an exact structure on $\text{maps}(\mathcal{C})$ we proceed as follows: we identify first \mathcal{C} with the category $\mathbf{p}(\mathcal{C})$ of projective objects of $\mathcal{A} = \text{mod}(\mathcal{C})$, in this way $\text{maps}(\mathcal{C})$ is equivalent to $\text{maps}(\mathbf{p}(\mathcal{C}))$ which is embedded in the abelian category $\mathcal{B} = \text{maps}(\mathcal{A})$. We can define an exact structure $(\text{maps}(\mathcal{C}), \mathcal{S})$ giving a subfunctor F of $\text{Ext}_{\mathcal{B}}^1(-, ?)$. Let $\Psi : P^0(\mathcal{A}) \rightarrow \text{maps}(\mathcal{C})$ be the functor given above and $\alpha : \text{maps}(\mathcal{C}) \rightarrow \text{maps}(\mathbf{p}(\mathcal{C}))$ the natural equivalence. Since Ψ is dense any object in $\text{maps}(\mathcal{C})$ is of the form $\Psi(P.)$ and we define $\text{Ext}_F^1(\alpha\Psi(P.), \alpha\Psi(Q.))$ as $\alpha\Psi(\text{Ext}_{\mathcal{C}-(\mathcal{A})}^1(P., Q.))$. We obtain the exact structure on $\text{maps}(\mathcal{C})$ using the identification α .

Once we have the exact structure on $\text{maps}(\mathcal{C})$ the definition of a relative tilting subcategory $\mathcal{T}_{\mathcal{C}}$ of $\text{maps}(\mathcal{C})$ is very natural, it will be equivalent to the following:

Definition 2.6. *A relative tilting category in the category of maps, $\text{maps}(\mathcal{C})$, is a subcategory $\mathcal{T}_{\mathcal{C}}$ such that :*

- (i) *Given $T : T_1 \rightarrow T_0$ in $\mathcal{T}_{\mathcal{C}}$, and $P. \in P^0(\mathcal{C})$ such that $\Psi(P.) = T$, there exist an integer n such that $\text{rpdim} P. \leq n$.*
- (ii) *Given $T : T_1 \rightarrow T_0$, $T' : T'_1 \rightarrow T'_0$ in $\mathcal{T}_{\mathcal{C}}$ and $\Psi(P.) = T$, $\Psi(Q.) = T'$, $P., Q. \in P^0(\text{mod}(\mathcal{C}))$. Then $\text{Ext}_{\mathcal{C}-(\text{mod}(\mathcal{C}))}^k(P., Q.) = 0$ for all $k \geq 1$.*
- (iii) *Given an object C in \mathcal{C} , denote by $(-, C)_{\circ}$ the complex $0 \rightarrow (-, C) \rightarrow 0$ concentrated in degree zero. Then there exists an exact sequence*

$$0 \rightarrow (-, C)_{\circ} \rightarrow P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_n \rightarrow 0$$

with $P_i \in P^0(\text{mod}(\mathcal{C}))$ and $\Psi(P_i) \in \mathcal{T}_{\mathcal{C}}$.

By definition, the following is clear

Theorem 2.7. *Let $\Phi : \text{maps}(\mathcal{C}) \rightarrow \text{mod}(\mathcal{C})$ be functor above, $\mathcal{T}_{\mathcal{C}}$ is a relative tilting subcategory of $\text{maps}(\mathcal{C})$ if and only if $\Phi(\mathcal{T}_{\mathcal{C}})$ is a tilting subcategory of $\text{mod}(\mathcal{C})$*

3. THE ALGEBRA OF TRIANGULAR MATRICES

Let Λ be an artin algebra. We want to explore the connections between $\text{mod } \Gamma$, with $\Gamma = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$ and the category $\text{mod}(\text{mod } \Lambda)$. In particular we want to compare the Auslander-Reiten quivers and subcategories which are tilting, contravariantly, covariantly and functorially finite. We identify $\text{mod } \Gamma$ with the category of Λ -maps, $\text{maps}(\Lambda)$ [see 7 Prop. 2.2]. We refer to the book by Fossum, Griffiths and Reiten [13] or to [16] for properties of modules over triangular matrix rings.

3.1. Almost Split Sequences. In this subsection we want to study the relation between the almost split sequences in $\text{mod } \Gamma$ and almost split sequences in $\text{mod}(\text{mod } \Lambda)$. We will see that except for a few special objects in $\text{mod } \Gamma$, the almost split sequences will belong to the class \mathcal{S} of the exact structure, so in particular will be relative almost split sequences.

For any indecomposable non projective Γ -module $M = (M_1, M_2, f)$ we can compute $DtrM$ as follows:

To construct a minimal projective resolution of M ([13], [16]), let $P_1 \xrightarrow{p_1} P_0 \rightarrow M_1 \rightarrow 0$ be a minimal projective presentation. Taking the cokernel, we have an exact sequence $M_1 \xrightarrow{f} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$, and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1 & \rightarrow & P_1 \oplus Q_1 & \rightarrow & Q_1 \longrightarrow 0 \\ & & p_1 \downarrow & & \downarrow & & q_1 \downarrow \\ 0 & \longrightarrow & P_0 & \rightarrow & P_0 \oplus Q_0 & \rightarrow & Q_0 \longrightarrow 0 \\ & & p_0 \downarrow & & \downarrow & & q_0 \downarrow \\ & & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{f_2} & M_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

with Q_0 the projective cover of M_3 . The presentation can be written as:

$$\begin{pmatrix} P_1 \\ P_1 \oplus Q_1 \end{pmatrix} \rightarrow \begin{pmatrix} P_0 \\ P_0 \oplus Q_0 \end{pmatrix} \rightarrow \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \rightarrow 0$$

and trM will look as follows:

$$\begin{pmatrix} P_0^* \oplus Q_0^* \\ Q_0^* \end{pmatrix} \rightarrow \begin{pmatrix} P_1^* \oplus Q_1^* \\ Q_1^* \end{pmatrix} \rightarrow tr \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \rightarrow 0$$

which corresponds to the commutative exact diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_0^* & \longrightarrow & Q_0^* \oplus P_0^* & \longrightarrow & P_0^* \longrightarrow 0 \\ & & q_1^* \downarrow & & \downarrow & & p_1^* \downarrow \\ 0 & \longrightarrow & Q_1^* & \longrightarrow & Q_1^* \oplus P_1^* & \longrightarrow & P_1^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & trM_3 \oplus Q^* & \rightarrow & trM_2 \oplus P^* & \longrightarrow & trM_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

with Q^*, P^* , projectives coming from the fact that the presentations of M_2 and M_3 in the first diagram are not necessary minimal.

Then τM is obtained as $\tau(M_1, M_2, f) = \tau M_2 \oplus D(P^*) \rightarrow \tau M_3 \oplus D(Q^*)$, with kernel $0 \rightarrow \tau M \rightarrow \tau M_2 \oplus D(P^*) \rightarrow \tau M_3 \oplus D(Q^*)$.

We consider now the special cases of indecomposable Γ -modules of the form: $M \xrightarrow{1_M} M$, $(M, 0, 0)$, $(0, M, 0)$, with M a non projective indecomposable Λ -module.

Proposition 3.1. *Let $0 \rightarrow \tau M \xrightarrow{j} E \xrightarrow{\pi} M \rightarrow 0$ be an almost split sequence of Λ -modules.*

(a) *Then the exact sequences of Γ -modules:*

- (1) $0 \rightarrow (\tau M, 0, 0) \xrightarrow{(j \ 0)} (E, M, \pi) \xrightarrow{(\pi \ 0)} (M, M, 1_M) \rightarrow 0,$
 (2) $0 \rightarrow (\tau M, \tau M, 1_{\tau M}) \xrightarrow{(1_{\tau M} \ j)} (\tau M, E, j) \xrightarrow{(0 \ \pi)} (0, M, 0) \rightarrow 0,$
are almost split.

- (b) *Given a minimal projective resolution $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \rightarrow 0$, we obtain a commutative diagram:*

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \tau M & \xrightarrow{j} & E & \xrightarrow{\pi} & M \longrightarrow 0 \\ & & \parallel & & \downarrow \bar{t} & & \downarrow t \\ 0 & \longrightarrow & \tau M & \xrightarrow{u} & D(P_1^*) & \xrightarrow{D(p_1^*)} & D(P_0^*) \end{array}$$

Then the exact sequence

$$0 \rightarrow (N_1, N_2, g) \xrightarrow{(j_2 \ j_1)} (E_1, E_2, h) \xrightarrow{(\pi_1 \ \pi_2)} (M_1, M_2, f) \rightarrow 0,$$

with $(N_1, N_2, g) = (D(P_1^), D(P_0^*), D(p_1^*))$, $(M_1, M_2, f) = (M, 0, 0)$,
 $(E_1, E_2, h) = (D(P_1^*) \oplus M, D(P_0^*), (D(p_1^*) \ t))$ and $(j_2 \ j_1) = (\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \ 1)$,
 $(\pi_1 \ \pi_2) = (0 \ -1)$, is an almost split sequence.*

Proof. (a) (1) Since $\pi : E \rightarrow M$ does not splits, the map $(\pi, 1_M) : (E, M, \pi) \rightarrow (M, M, 1_M)$ does not split. Let $(q_1, q_2) : (X_1, X_2, f) \rightarrow (M, M, 1_M)$ be a map that is not a splittable epimorphism. Then $q_2 f = q_1 1_M = q_1$.

We claim q_1 is not a splittable epimorphism. Indeed, if q_1 is a splittable epimorphism, then there exists a morphism $s : M \rightarrow X_1$, such that $q_1 s = 1_M$ and we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{1_M} & M \\ s \downarrow & & f s \downarrow \\ X_1 & \xrightarrow{f} & X_2 \\ q_1 \downarrow & & q_2 \downarrow \\ M & \xrightarrow{1_M} & M \end{array}$$

with $q_2 f s = q_1 s = 1_M$, and $(q_1, q_2) : (X_1, X_2, f) \rightarrow (M, M, 1_M)$ is a splittable epimorphism, a contradiction.

Since $\pi : E \rightarrow M$ is a right almost split morphism, there exists a map $h : X_1 \rightarrow E$ such that $\pi h = q_1$, and $q_2 f = q_1 = \pi h$. We have the following commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ h \downarrow & & q_2 \downarrow \\ E & \xrightarrow{\pi} & M \\ \pi \downarrow & & 1_M \downarrow \\ M & \xrightarrow{1_M} & M \end{array}$$

with $(\pi \ 1_M)(h \ q_2) = (q_1 \ q_2)$. We get a lifting $(h \ q_2) : (X_1, X_2, f) \rightarrow (E, M, \pi)$ of (q_1, q_2) . We have proved $\tau(M, M, 1_M) = (\tau M, 0, 0)$.

(2) It is clear, $\tau(0, M, 0) = (\tau M, \tau M, 1_{\tau M})$ and the exact sequence does not split. Now, let $(0, \rho) : (0, M, 0) \rightarrow (0, M, 0)$ be a non isomorphism. Then there exists a map $h : M \rightarrow E$ with $\pi h = \rho$. We have $(0 \ \pi)(0 \ h) = (0 \ \rho)$.

- (b) We have the following commutative diagram:

$$\begin{array}{ccc}
P_1 & \xrightarrow{\begin{pmatrix} 1_{P_1} \\ 0 \end{pmatrix}} & P_1 \oplus P_0 \xrightarrow{\begin{pmatrix} p_1 & 1_{P_0} \end{pmatrix}} P_0 \\
\downarrow p_1 & & \downarrow (p_1 \ 1_{P_0}) \\
P_0 & \xrightarrow{1_{P_0}} & P_0 \\
p_0 \downarrow & & \downarrow \\
M & \longrightarrow & 0 \\
\downarrow & & \\
0 & &
\end{array}$$

which implies the existence of the following commutative diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & P_0^* & \xrightarrow{1_{P_0^*}} & P_0^* \\
\downarrow & & \downarrow & & \downarrow p_1^* \\
P_0^* & \xrightarrow{\begin{pmatrix} 1_{P_0^*} \\ p_1^* \end{pmatrix}} & P_0^* \oplus P_1^* & \xrightarrow{\begin{pmatrix} 0 & 1_{P_1^*} \end{pmatrix}} & P_1^* \\
\downarrow 1_{P_0^*} & & \downarrow (p_1^* \ -1_{P_1^*}) & & \downarrow \\
P_0^* & \xrightarrow{p_1^*} & P_1^* & \longrightarrow & tr M \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

and $Dtr(M, 0, 0) = (D(P_1^*) \xrightarrow{D(p_1^*)} D(P_0^*))$. Since $j : \tau M \rightarrow E$ is a left almost split map, it extends to the map $\tau M \rightarrow D(P_1^*)$. We have the commutative diagram (3.1).

Hence, E is the pullback of the maps $t : M \rightarrow D(P_0^*)$, $D(P_1^*) \rightarrow D(P_0^*)$. We have an exact sequence:

$$0 \rightarrow E \xrightarrow{\begin{pmatrix} \tau \\ -\pi \end{pmatrix}} D(P_1^*) \oplus M \xrightarrow{(D(p_1^*) \ t)} D(P_0^*)$$

from which we built an exact commutative diagram:

$$\begin{array}{ccccc}
\tau M & \xrightarrow{u} & D(P_1^*) & \xrightarrow{D(p_1^*)} & D(P_0^*) \\
\downarrow j & & \downarrow \begin{pmatrix} 1_{D(P_1^*)} \\ 0 \end{pmatrix} & & \downarrow 1_{D(P_0^*)} \\
E & \xrightarrow{\begin{pmatrix} \tau \\ -\pi \end{pmatrix}} & D(P_1^*) \oplus M & \xrightarrow{(D(p_1^*) \ t)} & D(P_0^*) \\
\downarrow \pi & & \downarrow (0 \ -1_M) & & \downarrow \\
M & \xrightarrow{1_M} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

We claim that the exact sequence

$$0 \rightarrow (D(P_1^*), D(P_0^*), D(p_1^*)) \rightarrow (D(P_1^*) \oplus M, D(P_0^*), (D(p_1^*) t)) \rightarrow (M, 0, 0) \rightarrow 0$$

is an almost split sequence.

We need to prove first that it does not split. Suppose there exists a map $\begin{pmatrix} \mu \\ v \end{pmatrix} : (M, 0, 0) \rightarrow (D(P_1^*) \oplus M, D(P_0^*), (D(p_1^*) t))$ such that $((0 - 1_M) 0) \begin{pmatrix} \mu \\ v \end{pmatrix} = (1_M 0)$, then $(D(p_1^*) t) \begin{pmatrix} \mu \\ v \end{pmatrix} = 0$ and $v = -1_M$. It follows that there exists $s : M \rightarrow E$ such that $\begin{pmatrix} \bar{t} \\ -\pi \end{pmatrix} s = \begin{pmatrix} \mu \\ v \end{pmatrix}$, therefore $-\pi s = v = -1$, which implies $0 \rightarrow \tau M \rightarrow E \xrightarrow{\pi} M \rightarrow 0$ splits.

It will be enough to prove that any automorphism $(\sigma, 0) : (M, 0, 0) \rightarrow (M, 0, 0)$, which is not an isomorphism, lifts to $D(P_1^*) \oplus M \rightarrow D(P_0^*)$. But there exist a map $s : M \rightarrow E$ with $\pi s = \sigma$. We have a map

$$\begin{pmatrix} \bar{t} s \\ -\pi s \end{pmatrix} 0 : (M, 0, 0) \rightarrow (D(P_1^*) \oplus M, D(P_0^*), (D(p_1^*) t))$$

such that $((0 - 1_M) 0) \begin{pmatrix} \bar{t} s \\ -\pi s \end{pmatrix} 0 = (\sigma 0)$. \square

Dually, we consider almost split sequences of the form

$$0 \rightarrow (N_1, N_2, g) \xrightarrow{(j_2 \ j_1)} (E_1, E_2, h) \xrightarrow{(\pi_1 \ \pi_2)} (M_1, M_2, f) \rightarrow 0,$$

such that (N_1, N_2, g) is one of the following cases $(N, N, 1_N), (N, 0, 0), (0, N, 0)$, with N a non injective indecomposable Λ -module to have the following:

Proposition 3.2. *Let $0 \rightarrow N \xrightarrow{j} E \xrightarrow{\pi} \tau^{-1}N \rightarrow 0$ an almost split sequence of Λ -modules.*

(a) *Then the exact sequences of Γ -modules*

$$\begin{aligned} (1) \quad & 0 \rightarrow (N, N, 1_N) \xrightarrow{(1_N \ j)} (\tau M, E, j) \xrightarrow{(0 \ \pi)} (0, M, 0) \rightarrow 0 \\ (2) \quad & 0 \rightarrow (N, 0, 0) \xrightarrow{(j \ 0)} (E, \tau^{-1}N, \pi) \xrightarrow{(\pi \ 1)} (\tau^{-1}N, \tau^{-1}N, 1_{\tau^{-1}N}) \rightarrow 0 \\ & \text{are almost split.} \end{aligned}$$

(b) *Given a minimal injective resolution $0 \rightarrow N \xrightarrow{q_0} I_0 \xrightarrow{q_1} I_1$, we obtain a commutative diagram*

$$\begin{array}{ccccccc} D(I_0)^* & \xrightarrow{D(q_1)^*} & D(I_1)^* & \longrightarrow & \tau^{-1}N & \longrightarrow & 0 \\ v \downarrow & & \bar{v} \downarrow & & \parallel & & \\ 0 & \longrightarrow & N & \xrightarrow{j} & E & \xrightarrow{\pi} & \tau^{-1}N \longrightarrow 0 \end{array}$$

Then the exact sequence

$$0 \rightarrow (N_1, N_2, g) \xrightarrow{(j_2 \ j_1)} (E_1, E_2, h) \xrightarrow{(\pi_1 \ \pi_2)} (M_1, M_2, f) \rightarrow 0,$$

with $(N_1, N_2, g) = (0, N, 0)$, $(E_1, E_2, h) = (D(I_0)^, D(I_1)^* \oplus N, \begin{pmatrix} D(q_1)^* \\ v \end{pmatrix})$, $(M_1, M_2, f) = (D(I_0)^*, D(I_0)^*, D(q_1)^*)$ and $(j_2 \ j_1) = (0 \ \begin{pmatrix} 0 \\ 1 \end{pmatrix})$, $(\pi_1 \ \pi_2) = (1 \ \begin{pmatrix} 1 & 0 \end{pmatrix})$, is an almost split sequence.*

We will prove next that almost split sequences of objects which do not belong to the special cases consider before, are exact sequences in the relative structure \mathcal{S} .

Theorem 3.3. *Let*

$$0 \rightarrow (N_1, N_2, g) \xrightarrow{(j_1 \ j_2)} (E_1, E_2, h) \xrightarrow{(p_1 \ p_2)} (M_1, M_2, f) \rightarrow 0$$

be an almost split sequence of Γ -modules and assume that both g, f , are neither split-table epimorphisms, nor splittable monomorphisms. Consider the following commutative exact diagram:

$$(3.2) \quad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 \longrightarrow 0 \\ & & j_0 \downarrow & & j_1 \downarrow & & j_2 \downarrow & & j_3 \downarrow \\ 0 & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 \longrightarrow 0 \\ & & p_0 \downarrow & & p_1 \downarrow & & p_2 \downarrow & & p_3 \downarrow \\ 0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & 0 & & 0 & & 0 \end{array}$$

Then, the map $p_0 : E_0 \rightarrow M_0$ is an epimorphism, $j_3 : N_3 \rightarrow E_3$ is a monomorphism, and the exact sequences

$$(3.3) \quad 0 \rightarrow N_i \xrightarrow{j_i} E_i \xrightarrow{p_i} M_i \rightarrow 0, \quad 0 \leq i \leq 3$$

split.

Proof. The map $(1_{M_1} \ f) : (M_1, M_1, 1_{M_1}) \rightarrow (M_1, M_2, f)$ is not a splittable epimorphism. Therefore it factors through $E_1 \xrightarrow{h} E_2$. Hence; there exists a map $(t_1 \ t_2) : (M_1, M_2, f) \rightarrow (E_1, E_2, h)$, with $(p_1 \ p_2)(t_1 \ t_2) = (1_{M_1} \ f)$. Similarly, the map $(0 \ 1_{M_2}) : (0, M_2, 0) \rightarrow (M_1, M_2, f)$ is not a splittable epimorphism. Hence; there exists a map $(t_1 \ t_2) : (M_1, M_2, f) \rightarrow (E_1, E_2, h)$ with $(p_1 \ p_2)(t_1 \ t_2) = (0 \ 1_{M_2})$. We have proved that for $i = 1, 2$, the exact sequences $0 \rightarrow N_i \xrightarrow{j_i} E_i \xrightarrow{p_i} M_i \rightarrow 0$ split.

The diagram (3.2) induces the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (-, N_0) & \longrightarrow & (-, N_1) & \xrightarrow{(-, g)} & (-, N_2) & \xrightarrow{\tau} & G \longrightarrow 0 \\ & & (-, j_0) \downarrow & & (-, j_1) \downarrow & & (-, j_2) \downarrow & & \rho \downarrow \\ 0 & \longrightarrow & (-, E_0) & \longrightarrow & (-, E_1) & \xrightarrow{(-, h)} & (-, E_2) & \xrightarrow{\pi} & H \longrightarrow 0 \\ & & (-, p_0) \downarrow & & (-, p_1) \downarrow & & (-, p_2) \downarrow & & \sigma \downarrow \\ 0 & \longrightarrow & (-, M_0) & \longrightarrow & (-, M_1) & \xrightarrow{(-, f)} & (-, M_2) & \xrightarrow{\eta} & F \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & 0 & & 0 & & 0 \end{array}$$

By the Snake's Lemma, we have a connecting map δ ,

$$\cdots \rightarrow (-, E_0) \xrightarrow{(-, p_0)} (-, M_0) \xrightarrow{\delta} G \xrightarrow{\rho} H \rightarrow \cdots$$

We want to prove ρ is a monomorphism. Let $\rho : G \xrightarrow{\rho_1} \text{Im} \rho \xrightarrow{\rho_2} H$ be a factorization through its image.

Since $\text{mod}(\text{mod} \Lambda)$ is an abelian category, $\text{Im} \rho$ is a finitely presented functor, with presentation

$$(-, X_1) \xrightarrow{(-, t)} (-, X_2) \rightarrow \text{Im} \rho \rightarrow 0.$$

Lifting the maps ρ_1, ρ_2 we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc} (-, N_1) & \xrightarrow{(-, g)} & (-, N_2) & \longrightarrow & G & \longrightarrow & 0 \\ (-, u_1) \downarrow & & (-, u_2) \downarrow & & \rho_1 \downarrow & & \\ (-, X_1) & \xrightarrow{(-, t)} & (-, X_2) & \longrightarrow & \text{Im} \rho & \longrightarrow & 0 \\ (-, v_1) \downarrow & & (-, v_2) \downarrow & & \rho_1 \downarrow & & \\ (-, E_1) & \xrightarrow{(-, h)} & (-, E_2) & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

whose composition is another lifting of ρ . Then the two liftings are homotopic and there exist maps $(-, s_1) : (-, N_2) \rightarrow (-, E_1)$, $(-, s_2) : (-, N_1) \rightarrow (-, E_0)$ such that $(-, j_2) = (-, h)(-, s_1) + (-, v_1 u_1)$, $(-, j_2) = (-, h_1)(-, s_2) + (-, s_1)(-, g) + (-, v_2 u_2)$. This is $j_2 = h s_1 + v_1 u_1$, $j_1 = h_1 s_2 + s_1 g + v_2 u_2$. Consider the following commutative diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{g} & N_2 \\ \left(\begin{smallmatrix} u_1 \\ s_1 g \\ s_2 \end{smallmatrix} \right) \downarrow & \left(\begin{smallmatrix} t & 0 & 0 \\ 0 & 1_{E_1} & 0 \end{smallmatrix} \right) \downarrow & \left(\begin{smallmatrix} u_2 \\ s_1 \end{smallmatrix} \right) \downarrow \\ X_1 \oplus E_1 \oplus E_0 & \longrightarrow & X_2 \oplus E_1 \\ (v_1 \ 1_{E_1} \ h_1) \downarrow & & \downarrow (v_2 \ h) \\ E_1 & \xrightarrow{h} & E_2 \end{array}$$

But

$$(3.4) \quad (-, X_1 \oplus E_1 \oplus E_0) \xrightarrow{\left(-, \left(\begin{smallmatrix} t & 0 & 0 \\ 0 & 1_{E_1} & 0 \end{smallmatrix} \right) \right)} (-, X_2 \oplus E_1) \rightarrow \text{Im} \rho \rightarrow 0$$

is exact. Changing $(-, X_1) \xrightarrow{(-, t)} X_2$ by (3.4), we can assume $v_1 u_1 = l_i$, $i = 1, 2$; but being $(l_1, l_2) : (N_1, N_2, g) \rightarrow (E_1, E_2, h)$ an irreducible map, this implies either $(u_1, u_2) : (N_1, N_2, g) \rightarrow (X_1, X_2, t)$ is a splittable monomorphism or $(v_1, v_2) : (X_1, X_2, t) \rightarrow (E_1, E_2, h)$ is a splittable epimorphism.

In the second case we have a map $(s_1, s_2) : (E_1, E_2, h) \rightarrow (X_1, X_2, t)$, with $(v_1 \ v_2)(s_1 \ s_2) = (1_{E_1} \ 1_{E_2})$. Then there exists a map $\sigma : H \rightarrow \text{Im} \rho$, such that $\rho_2 \sigma = 1_H$. It follows ρ_2 is an isomorphism. Hence; $F = 0$ and $f : M_1 \rightarrow M_2$ is a splittable epimorphism. A contradiction.

Now, if $(u_1 \ u_2)$ is a splittable monomorphism, then there exists a map $(q_1 \ q_2) : (X_1, X_2, t) \rightarrow (N_1, N_2, g)$, with $(q_1 \ q_2)(u_1 \ u_2) = (1_{N_1} \ 1_{N_2})$. Then, there exists $\sigma : \text{Im} \rho \rightarrow G$ such that $\sigma \rho = 1_G$, and ρ_1 is an isomorphism, in particular ρ is a

monomorphism. It follows $(-, E_0) \xrightarrow{(-, p_0)} (-, M_0)$ is an epimorphism. Therefore: $0 \rightarrow N_0 \xrightarrow{j_0} E_0 \xrightarrow{p_0} M_0 \rightarrow 0$ is an exact sequence that splits. A contradiction.

Dualizing the diagram, we obtain, the exact sequence $0 \rightarrow D(M_3) \rightarrow D(E_3) \rightarrow D(N_3) \rightarrow 0$ splits. Therefore the exact sequence $0 \rightarrow N_3 \rightarrow E_3 \rightarrow M_3 \rightarrow 0$ splits. \square

We can see now that the functor Φ preserves almost split sequences.

Theorem 3.4. *Let*

$$0 \rightarrow (N_1, N_2, g) \xrightarrow{(j_1 \ j_2)} (E_1, E_2, h) \xrightarrow{(p_1 \ p_2)} (M_1, M_2, f) \rightarrow 0$$

be an almost split sequence, such that g, f , are neither splittable epimorphisms nor splittable monomorphisms. Then the exact sequence

$$0 \rightarrow G \xrightarrow{\rho} H \xrightarrow{\theta} F \rightarrow 0$$

obtained from the commutative diagram:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ (-, N_1) & \xrightarrow{(-, g)} & (-, N_2) & \longrightarrow & G \longrightarrow 0 \\ \downarrow (-, j_1) & & \downarrow (-, j_2) & & \downarrow \rho \\ (-, E_1) & \xrightarrow{(-, h)} & (-, E_2) & \longrightarrow & H \longrightarrow 0 \\ \downarrow (-, p_1) & & \downarrow (-, p_2) & & \downarrow \theta \\ (-, M_1) & \xrightarrow{(-, f)} & (-, M_2) & \longrightarrow & F \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

is an almost split sequence

Proof. (1) The sequence $0 \rightarrow G \xrightarrow{\rho} H \xrightarrow{\theta} F \rightarrow 0$ does not split.

Assume it does split and let $u : F \rightarrow H$, with $\theta u = 1_F$ be the splitting. There is a lifting of u making the following diagram, with exact rows, commute:

$$\begin{array}{ccccccc} (-, M_0) & \xrightarrow{(-, f_1)} & (-, M_1) & \xrightarrow{(-, f)} & (-, M_2) & \longrightarrow & F \longrightarrow 0 \\ \downarrow (-, s_0) & & \downarrow (-, s_1) & & \downarrow (-, s_2) & & \downarrow \\ (-, E_0) & \xrightarrow{(-, h_1)} & (-, E_1) & \xrightarrow{(-, h)} & (-, E_2) & \longrightarrow & H \longrightarrow 0 \\ \downarrow (-, p_0) & & \downarrow (-, p_1) & & \downarrow (-, p_2) & & \downarrow \\ (-, M_0) & \xrightarrow{(-, f_1)} & (-, M_1) & \xrightarrow{(-, f)} & (-, M_2) & \longrightarrow & F \longrightarrow 0 \end{array}$$

The composition is a lifting of the identity, and as before, it is homotopic to the identity. By Yoneda's lemma, there exist maps, $w_2 : M_2 \rightarrow M_1$, $w_1 : M_1 \rightarrow M_0$, such that $f w_2 + p_2 s_2 = 1_{M_2}$, $w_2 f + f_1 w_1 + p_1 s_1 = 1_{M_1}$. Since $f \in \text{rad}(M_1, M_2)$, $f_1 \in \text{rad}(M_0, M_1)$, this implies $w_2 f, f_1 w_1 \in \text{radEnd}(M_1)$ and $f w_2 \in \text{radEnd}(M_2)$. It follows $p_2 s_2 = 1_{M_2} - f w_2$ and $p_1 s_1 = 1_{M_1} - (w_2 f + f_1 w_1)$ are invertible. Therefore: $(p_1 \ p_2) : (E_1, E_2, h) \rightarrow (M_1, M_2, f)$ is a splittable epimorphism. A contradiction.

(2) Let $\eta : L \rightarrow F$ be a non splittable epimorphism, and $(-, X_1) \xrightarrow{(-, t)} (-, X_2) \rightarrow L \rightarrow 0$ a projective presentation of L . The map η lifts to a map $(-, \eta_i) : (-, X_i) \rightarrow (-, M_i)$, $i = 1, 2$. Then the map $(\eta_1 \ \eta_2) : (X_1, X_2, t) \rightarrow (M_1, M_2, f)$ is not a splittable epimorphism., and there exists a map $(v_1 \ v_2) : (X_1, X_2, t) \rightarrow (E_1, E_2, h)$, with $(p_1 \ p_2)(v_1 \ v_2) = (\eta_1 \ \eta_2)$.

The map $(v_1 \ v_2)$ induces a map $\sigma : L \rightarrow H$ with $\theta\sigma = \eta$.

In a similar way we prove $0 \rightarrow G \rightarrow H$ is left almost split. \square

Assume now (*) $0 \rightarrow G \rightarrow H \rightarrow F \rightarrow 0$ is an almost split sequence in $\text{mod}(\text{mod}\Lambda)$. Let $(-, M_1) \xrightarrow{(-, f)} (-, M_2) \rightarrow F \rightarrow 0$ be a minimal projective projective presentation of F . The map $M_1 \xrightarrow{f} M_2$ is an indecomposable object in $\text{mod}\begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$ and is not projective.

Then we have an almost split sequence in $\text{maps}(\Lambda)$:

$$0 \rightarrow N = (N_1, N_2, g) \xrightarrow{(j_1 \ j_2)} E = (E_1, E_2, h) \xrightarrow{(p_1 \ p_2)} M = (M_1, M_2, f) \rightarrow 0$$

where f, g are both neither splittable monomorphisms and nor splittable epimorphisms. Applying the functor Φ to the above sequence we obtain an almost split sequence (**) $0 \rightarrow \Phi(N) \rightarrow \Phi(E) \rightarrow \Phi(M) \rightarrow 0$ with $\Phi(M) = F$. By the uniqueness of the almost split sequence $\Phi(N) = G$, $\Phi(E) = H$ and (*) is isomorphic to (**).

3.1.1. *An example.* Let $\Lambda = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$ be the algebra isomorphic to the quiver algebra KQ , where Q is: $1 \rightarrow 2$ and $\Gamma = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$. The algebra Λ has a simple projective S_2 , a simple injective S_1 and a projective injective P_1 .

The projective Γ -modules correspond to the maps: $0 \rightarrow S_2$, $0 \rightarrow P_1$, $P_1 \xrightarrow{1_{P_1}} P_1$ and $S_2 \xrightarrow{1_{S_2}} S_2$. We compute the almost split sequences in $\text{maps}(\text{mod}(\Lambda))$ to obtain exact sequences:

$$0 \rightarrow N \xrightarrow{j} E \xrightarrow{\pi} M \rightarrow 0$$

with:

(a) $N = (0, S_2, 0)$, $E = (S_2, S_2 \oplus P_1, \begin{pmatrix} 1_{S_2} \\ 0 \end{pmatrix})$, $M = (S_2, P_1, f)$, $j = (0, \begin{pmatrix} 1_{S_2} \\ f \end{pmatrix})$, $\pi = (1_{S_2}, (f, -1_{P_1}))$

(b) $N = (S_2, P_1, f)$, $E = (P_1 \oplus S_2, P_1 \oplus S_1, \begin{pmatrix} 1_{P_1} & 0 \\ 0 & 0 \end{pmatrix})$, $M = (P_1, S_1, g)$, $j = (\begin{pmatrix} f \\ 1_{S_2} \end{pmatrix}, \begin{pmatrix} 1_{P_1} \\ g \end{pmatrix})$, $\pi = ((-1_{P_1}, f), (-g, 1_{S_1}))$.

(c) $N = (P_1, S_1, g)$, $E = (S_1 \oplus P_1, S_1, (1_{S_1} \ 0))$, $M = (S_1, 0, 0)$, $j = (\begin{pmatrix} g \\ 1_{P_1} \end{pmatrix} \ 1_{S_1})$, $\pi = ((-1_{S_1} \ g) \ 0)$.

The Auslander-Reiten quiver in $\text{maps}(\Lambda)$ is:

$$\begin{array}{ccccccc} & & (S_2, S_2, 1) & & (P_1, P_1, 1) & & (S_1, S_1, 1) \\ & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow \\ (0, S_2, 0) & \xrightarrow{\quad} & (S_2, P_1, f) & \xrightarrow{\quad} & (S_2, 0, 0) & \xrightarrow{\quad} & (P_1, S_1, g) & \xrightarrow{\quad} & (S_1, 0, 0) \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \\ & & (0, P_1, 0) & & (0, S_1, 0) & & (P_1, 0, 0) & & \end{array}$$

Applying the functor Φ we obtain the following Auslander-Reiten quiver of $\text{mod}(\text{mod}\Lambda)$:

$$0 \rightarrow (-, S_2) \rightarrow (-, P_1) \rightarrow \text{rad}(-, S_1) \rightarrow (-, S_1) \rightarrow S_{S_1} \rightarrow 0,$$

which is isomorphic to the Auslander-Reiten quiver of $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, with $\beta\alpha = 0$, and this is the Auslander algebra of Λ .

3.2. Tilting in $\text{mod}(\text{mod}\Lambda)$. Let Λ be an artin algebra. Since $\text{maps}(\text{mod}\Lambda)$ is equivalent to the category $\text{mod } \Gamma$, with $\Gamma = \begin{pmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{pmatrix}$, it is abelian, dualizing Krull-Schmidt, and it has kernels. Hence it has pseudokernels, and we can apply the theory so far developed. In this case the exact structure is easy to describe.

The collection of exact sequences \mathcal{S} consists of the exact sequences in the category $\text{maps}(\text{mod}\Lambda)$

$$0 \rightarrow (N_1, N_2, g) \rightarrow (E_1, E_2, h) \rightarrow (M_1, M_2, f) \rightarrow 0$$

such that in the following exact commutative diagram

$$(3.5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & N_0 & \xrightarrow{g_0} & N_1 & \xrightarrow{g} & N_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & E_0 & \xrightarrow{h_0} & E_1 & \xrightarrow{h} & E_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & M_0 & \xrightarrow{f_0} & M_1 & \xrightarrow{f} & M_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

the columns split. Here (N_0, g_0) , (E_0, h_0) , (M_0, f_0) are the kernels of the maps g , h and f , respectively.

The collection \mathcal{S} gives rise to a subfunctor F of the additive bifunctor $\text{Ext}_\Gamma^1(-, -) : (\text{mod}\Gamma) \times (\text{mod}\Gamma)^{op} \rightarrow \mathbf{Ab}$. The category $\text{maps}(\text{mod}\Lambda)$ has enough F -projectives and enough F -injectives, the F -projectives are the maps of the form $M \xrightarrow{1_M} M$ and $0 \rightarrow M$, and the F -injectives are of the maps of the form $M \xrightarrow{1_M} M$ and $M \rightarrow 0$.

According to Theorem 2.7 we have the following:

Theorem 3.5. *Classical tilting subcategories in $\text{mod}(\text{mod}\Lambda)$ correspond under Ψ with relative tilting subcategories $\mathcal{T}_{\text{mod}\Lambda}$ of $\text{maps}(\text{mod}\Lambda)$, such that the following statements hold:*

- (i) *The maps $f : T_0 \xrightarrow{f} T_1$ of objects in $\mathcal{T}_{\text{mod}\Lambda}$ are monomorphisms.*
- (ii) *Given $T : f : T_0 \xrightarrow{f} T_1$ and $T' : g : T'_0 \xrightarrow{g} T'_1$ in $\mathcal{T}_{\mathcal{C}}$. Then $\text{Ext}_F^1(T, T') = 0$.*
- (iii) *For each object C in \mathcal{C} , there exist a exact sequence in $\text{maps}(\text{mod}\Lambda)$:*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C \\
\downarrow & & \downarrow \\
T_0 & \xrightarrow{f} & T_1 \\
\downarrow & & \downarrow \\
T_0 & \xrightarrow{g} & T'_1 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

such that the second column splits and $T : f : T_0 \rightarrow T_1$, $T' : g : T_0 \rightarrow T'_1$ are in \mathcal{T}_C .

Since $\text{gdim}(\text{mod } \Lambda) \leq 2$, the relative global dimension of $\text{maps}(\text{mod } \Lambda)$ is ≤ 2 .

For generalized tilting subcategories of $\text{maps}(\text{mod } \Lambda)$ there is the following analogous to the previous theorem:

Theorem 3.6. *Generalized tilting subcategories of $\text{mod}(\text{mod } \Lambda)$ correspond under Ψ with relative tilting subcategories $\mathcal{T}_{\text{mod } \Lambda}$ of $\text{maps}(\text{mod } \Lambda)$ such that the following statements hold:*

- (i) *Given $T : T_1 \rightarrow T_0$, $T' : T'_1 \rightarrow T'_0$ in $\mathcal{T}_{\text{mod } \Lambda}$. Then $\text{Ext}_F^k(T, T') = 0$, for $0 < k \leq 2$.*
- (ii) *For each object C in \mathcal{C} , there exists a relative exact sequence in $\text{maps}(\text{mod } \Lambda)$:*

$$0 \rightarrow (0, C, 0) \rightarrow T^0 \rightarrow T^1 \rightarrow T^2 \rightarrow 0$$

with $T^i \in \mathcal{T}_{\text{mod } \Lambda}$.

3.3. Contravariantly Finite Categories in $\text{mod}(\text{mod } \Lambda)$. In this subsection we will see that some properties like: contravariantly, covariantly, functorially finite subcategories of $\text{maps}(\text{mod } \Lambda)$, are preserved by the functor Φ .

The following theorem was proved in [18]. [See also 4 Theo. 5.5].

Theorem 3.7. *Let \mathcal{C} be a dualizing Krull-Schmidt variety. The assignments $\mathcal{T} \mapsto \mathcal{T}^\perp$ and $\mathcal{Y} \mapsto \mathcal{Y} \cap^\perp \mathcal{Y}$ induce a bijection between equivalence classes of tilting subcategories \mathcal{T} of $\text{mod}(\mathcal{C})$, with $\text{pdim } \mathcal{T} \leq n$, such that \mathcal{T} is a generator of \mathcal{T}^\perp and classes of subcategories \mathcal{Y} of $\text{mod}(\mathcal{C})$ which are covariantly finite, coresolving, and whose orthogonal complement ${}^\perp \mathcal{Y}$ has projective dimension $\leq n$.*

Of course, the dual of the above theorem is true. Hence it is clear the importance of studying; covariantly, contravariantly and functorially finite subcategories in $\text{mod}(\mathcal{C})$. We are specially interested in the case \mathcal{C} is the category of finitely generated left modules over an artin algebra Λ . In this situation we can study them via the functor Ψ relating them with the corresponding subcategories of $\text{maps}(\text{mod } \Lambda)$, which in principle are easier to study, since $\text{maps}(\text{mod } \Lambda)$ and the category of finitely generated left Γ -modules, with Γ the triangular matrix ring, are equivalent.

Such is the content of our next theorem.

Theorem 3.8. *Let $\mathcal{C} \subset \text{maps}(\text{mod } \Lambda)$ be a subcategory. Then the following statements hold:*

- (a) If \mathcal{C} is contravariantly finite in $\text{maps}(\text{mod}\Lambda)$, then $\Phi(\mathcal{C})$ is a contravariantly finite subcategory of $\text{mod}(\text{mod}\Lambda)$.
- (b) If \mathcal{C} is covariantly finite in $\text{maps}(\text{mod}\Lambda)$, then $\Phi(\mathcal{C})$ is a covariantly finite subcategory of $\text{mod}(\text{mod}\Lambda)$.
- (c) If \mathcal{C} is functorially finite in $\text{maps}(\text{mod}\Lambda)$, then $\Phi(\mathcal{C})$ is a functorially finite subcategory of $\text{mod}(\text{mod}\Lambda)$.

Proof. (a) Assume $\mathcal{C} \subset \text{maps}(\text{mod}\Lambda)$ is contravariantly finite. Let F be a functor in $\text{mod}(\text{mod}\Lambda)$ and $(, M_1) \xrightarrow{(\cdot, f)} (, M_2) \rightarrow F \rightarrow 0$ a minimal projective presentation of F . Then, there exist a map $Z : Z_1 \xrightarrow{h} Z_2$ and a map

$$\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ q_1 \downarrow & & q_2 \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

in $\text{maps}(\text{mod}\Lambda)$ such that $Z = (Z_1, Z_2, h)$ is a right \mathcal{C} -approximation of $M = (M_1, M_2, f)$. The diagram (3.3) induce the following commutative exact diagram:

$$\begin{array}{ccccccc} (, Z_1) & \xrightarrow{(\cdot, h)} & (, Z_2) & \longrightarrow & \Phi(Z) & \longrightarrow & 0 \\ (\cdot, q_1) \downarrow & & (\cdot, q_2) \downarrow & & \rho \downarrow & & \\ (, M_1) & \xrightarrow{(\cdot, f)} & (, M_2) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

We claim that ρ is a right $\Phi(\mathcal{C})$ -approximation of F . Let $H \in \Phi(\mathcal{C})$, $\eta : H \rightarrow F$ a map and $(, X_1) \xrightarrow{(\cdot, r)} (, X_2) \rightarrow H \rightarrow 0$ a minimal projective presentation of H . We have a lifting of η :

$$\begin{array}{ccccccc} (, X_1) & \xrightarrow{(\cdot, r)} & (, X_2) & \longrightarrow & H & \longrightarrow & 0 \\ (\cdot, s_1) \downarrow & & (\cdot, s_2) \downarrow & & \eta \downarrow & & \\ (, M_1) & \xrightarrow{(\cdot, f)} & (, M_2) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

By Yoneda's Lemma, there is the following commutative square:

$$\begin{array}{ccc} X_1 & \xrightarrow{r} & X_2 \\ s_1 \downarrow & & s_2 \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

with $X = (X_1, X_2, r) \in \mathcal{C}$. Since $Z = (Z_1, Z_2, h)$ is a right \mathcal{C} -approximation of $M = (M_1, M_2, f)$, there exists a morphism $(t_1, t_2) : (X_1, X_2, r) \rightarrow (Z_1, Z_2, h)$, such that the following diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{r} & X_2 \\ t_1 \downarrow & & t_2 \downarrow \\ Z_1 & \xrightarrow{h} & Z_2 \\ q_1 \downarrow & & q_2 \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is commutative, with $q_i t_i = s_i$ for $i = 1, 2$. Which implies the existence of a map $\theta : H \rightarrow \Psi(H) = G$, such that the diagram

$$\begin{array}{ccccccc} (\cdot, X_1) & \xrightarrow{(\cdot, r)} & (\cdot, X_2) & \longrightarrow & H & \longrightarrow & 0 \\ (\cdot, t_1) \downarrow & & (\cdot, t_2) \downarrow & & \theta \downarrow & & \\ (\cdot, Z_1) & \xrightarrow{(\cdot, h)} & (\cdot, Z_2) & \longrightarrow & G & \longrightarrow & 0 \\ (\cdot, q_1) \downarrow & & (\cdot, q_2) \downarrow & & \rho \downarrow & & \\ (\cdot, M_1) & \xrightarrow{(\cdot, f)} & (\cdot, M_2) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

with exact rows, is commutative, this is: $\rho\theta = \eta$.

The proof of (b) is dual to (a), and (c) follows from (a) and (b). \square

Theorem 3.9. *Let $\mathcal{C} \subset \text{maps}(\text{mod}\Lambda)$ be a category which contains the objects of the form $(M, 0, 0)$, $(M, M, 1_M)$, and assume $\Phi(\mathcal{C})$ is contravariantly finite. Then \mathcal{C} is contravariantly finite.*

Proof. Let $M_1 \xrightarrow{f} M_2$ a map in $\text{maps}(\text{mod}\Lambda)$, then we have an exact sequence $(\cdot, M_1) \xrightarrow{(\cdot, f)} (\cdot, M_2) \rightarrow F \rightarrow 0$.

There exist $G \in \Phi(\mathcal{C})$ such that $\rho : G \rightarrow F$ is a right $\Phi(\mathcal{C})$ -approximation.

Let

$$(\cdot, Z_1) \xrightarrow{(\cdot, h)} (\cdot, Z_2) \rightarrow G \rightarrow 0$$

be a minimal projective presentation of G . Then, there exists a map $(r_1, r_2) : (Z_1, Z_2, h) \rightarrow (M_1, M_2, f)$ such that

$$\begin{array}{ccccccc} (\cdot, Z_1) & \xrightarrow{(\cdot, h)} & (\cdot, Z_2) & \longrightarrow & G & \longrightarrow & 0 \\ (\cdot, r_1) \downarrow & & (\cdot, r_2) \downarrow & & \rho \downarrow & & \\ (\cdot, M_1) & \xrightarrow{(\cdot, f)} & (\cdot, M_2) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

is a lifting of ρ .

Let (X_1, X_2, g) be an object in \mathcal{C} and a map $(v_1, v_2) : (X_1, X_2, g) \rightarrow (M_1, M_2, f)$, which induces the following commutative exact diagram in $\text{mod}(\text{mod}\Lambda)$:

$$\begin{array}{ccccccc} (\cdot, X_1) & \xrightarrow{(\cdot, g)} & (\cdot, X_2) & \longrightarrow & H & \longrightarrow & 0 \\ (\cdot, v_1) \downarrow & & (\cdot, v_2) \downarrow & & \eta \downarrow & & \\ (\cdot, M_1) & \xrightarrow{(\cdot, f)} & (\cdot, M_2) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

Since $\rho : G \rightarrow F$ is a right $\Phi(\mathcal{C})$ -approximation, there exists a morphism $\theta : H \rightarrow G$ such that $\rho\theta = \eta$. Therefore: θ induces a morphism

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ t_1 \downarrow & & t_2 \downarrow \\ Z_1 & \xrightarrow{h} & Z_2 \end{array}$$

in $\text{maps}(\mathcal{C})$ such that $\Phi(t_1, t_2) = \theta$.

We have two liftings of ρ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\ , X_0) & \xrightarrow{g_0} & (\ , X_1) & \xrightarrow{(\ , g)} & (\ , X_2) \longrightarrow H \longrightarrow 0 \\ & & (\ , v_0) \downarrow & & (\ , r_0 t_0) \downarrow & & (\ , v_1) \downarrow & & (\ , r_1 t_1) \downarrow & & (\ , v_2) \downarrow & & (\ , r_2 t_2) \downarrow & & \rho \downarrow \\ 0 & \longrightarrow & (\ , M_0) & \xrightarrow{f_0} & (\ , M_1) & \xrightarrow{(\ , f)} & (\ , M_2) \longrightarrow F \longrightarrow 0 \end{array}$$

Then they are homotopic, and there exist maps $(\ , \lambda_1) : (\ , X_1) \rightarrow (\ , M_0)$ and $(\ , \lambda_2) : (\ , X_2) \rightarrow (\ , M_1)$, such that

$$\begin{aligned} v_1 &= r_1 t_1 + f_0 \lambda_1 + \lambda_2 g, \\ v_2 &= r_2 t_2 + f \lambda_2. \end{aligned}$$

We have the following commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ m_1 \downarrow & & \downarrow m_2 \\ Z_1 \coprod M_0 \coprod M_1 & \xrightarrow{w} & Z_2 \coprod M_1 \\ n_1 \downarrow & & \downarrow n_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

with morphisms $n_1 = (r_1 \ f_0 \ 1_{M_1})$, $n_2 = (r_2 \ f_2)$ and

$$m_1 = \begin{pmatrix} t_1 \\ \lambda_1 \\ \lambda_2 g \end{pmatrix}, m_2 = \begin{pmatrix} t_2 \\ \lambda_2 \end{pmatrix}, w = \begin{pmatrix} h & 0 & 0 \\ 0 & 0 & 1_{M_1} \end{pmatrix}.$$

But $w : Z_1 \coprod M_0 \coprod M_1 \rightarrow Z_2 \coprod M_1$ is in \mathcal{C} , and

$$\begin{array}{ccc} Z_1 \coprod M_0 \coprod M_1 & \xrightarrow{w} & Z_2 \coprod M_1 \\ n_1 \downarrow & & \downarrow n_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is a right \mathcal{C} -approximation of $M_1 \xrightarrow{f} M_2$. □

We can define the functor $\Phi^{op} : \text{maps}(\text{mod } \Lambda) \rightarrow \text{mod}((\text{mod } \Lambda)^{op})$ as:

$$\Phi^{op}(A_1 \xrightarrow{f} A_0) = \text{Coker}((A_0, -) \xrightarrow{(f, -)} (A_1, -))$$

We have the following dual of the above theorem, whose proof we leave to the reader:

Theorem 3.10. *Let $\mathcal{C} \subset \text{maps}(\text{mod } \Lambda)$ be a subcategory that contains the objects of the form $(0, M, 0)$ and $(M, M, 1_M)$. If $\Phi^{op}(\mathcal{C})$ is contravariantly finite, then \mathcal{C} is covariantly finite.*

Definition 3.11. *The subcategory \mathcal{C} of $\text{maps}(\text{mod } \Lambda)$ consisting of all maps (M_1, M_2, f) , such that f is an epimorphism, will be called the category of epimaps, $\text{epimaps}(\text{mod } \Lambda)$. Dually, the subcategory \mathcal{C} of $\text{maps}(\text{mod } \Lambda)$ consisting of all maps (M_1, M_2, f) , such that f is a monomorphism, will be called the category of monomaps, $\text{monomaps}(\text{mod } \Lambda)$.*

We have the following examples of functorially finite subcategories of the category $\text{maps}(\text{mod}\Lambda)$:

Proposition 3.12. *The categories $\text{epimaps}(\text{mod}\Lambda)$ and $\text{monomaps}(\text{mod}\Lambda)$ are functorially finite in $\text{maps}(\text{mod}\Lambda)$.*

Proof. Let $M_1 \xrightarrow{f} M_2$ be an object in $\text{maps}(\text{mod}\Lambda)$. Then we have the following right approximation

$$\begin{array}{ccc} M_1 & \xrightarrow{f'} & \text{Im}(f) \longrightarrow 0 \\ \parallel & & \downarrow j \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Consider an epimorphism $X_2 \xrightarrow{g} X_2 \rightarrow 0$ and a map in $\text{maps}(\text{mod}\Lambda)$ $(t_1, t_2) : (X_1, X_2, g) \rightarrow (M_1, M_2, f)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{g} & X_2 & \longrightarrow & 0 \\ t_1 \downarrow & & t_2 \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 & \xrightarrow{\pi} & \text{Coker}(f) \longrightarrow 0 \end{array}$$

Since $\pi t_2 = 0$, the map $t_2 : X_2 \rightarrow M_2$ factors through $j : \text{Im}(f) \rightarrow M_2$, this is: there is a map $u : X_2 \rightarrow \text{Im}(f)$ such that $ju = t_2$, and we have the commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \longrightarrow 0 \\ t_1 \downarrow & & u \downarrow \\ M_1 & \xrightarrow{f'} & \text{Im}(f) \\ \parallel & & \downarrow j \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Now, let $P \xrightarrow{p} M_2 \rightarrow 0$ be the projective cover of M_2 . We get a commutative exact diagram:

$$(3.6) \quad \begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow \scriptstyle (1_0) & & \downarrow \scriptstyle 1_{M_2} \\ M_1 \oplus P & \xrightarrow{[f \ p]} & M_2 \longrightarrow 0 \end{array}$$

Let $X_1 \xrightarrow{g} X_2 \rightarrow 0$ be an epimorphism and

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ s_1 \downarrow & & \downarrow s_2 \\ X_1 & \xrightarrow{g} & X_2 \longrightarrow 0 \end{array}$$

a map of maps.

Since P is projective, we have the following commutative square

$$\begin{array}{ccc} P & \xrightarrow{p} & M_2 \\ \mu \downarrow & & s_2 \downarrow \\ X_1 & \xrightarrow{g} & X_2 \longrightarrow 0 \end{array}$$

Then gluing the two squares:

$$\begin{array}{ccccc} M_1 & \xrightarrow{f} & M_2 & & \\ \textstyle \binom{1}{0} \downarrow & & \parallel & & \\ M_1 \oplus P & \xrightarrow{[f \ p]} & M_2 & \longrightarrow & 0 \\ [s_1 \ \mu] \downarrow & & s_2 \downarrow & & \\ X_1 & \xrightarrow{g} & X_2 & \longrightarrow & 0 \end{array}$$

we obtain the map (s_1, s_2) .

Then (3.6) is a left approximation. The second part is dual. \square

Corollary 3.13. (i) *The category $\text{mod}(\Lambda)^O$ of functors vanishing on projectives is functorially finite.*
(ii) *The category $\text{mod}\hat{\Lambda}$ of functors with $\text{pd} \leq 1$ is functorially finite.*

Proof. The proof of this follows immediately from

$$\text{mod}(\Lambda)^O = \Phi(\text{epimaps}(\text{mod}\Lambda), \text{mod}\hat{\Lambda}) = \Phi(\text{monomaps}(\text{mod}\Lambda)).$$

\square

In view of the previous theorem it is of special interest to characterize the functors in $\text{mod}(\text{mod}\Lambda)$ of projective dimension less or equal to one.

The radical $t_H(F)$ of a finitely presented functor F , is defined as $t_H(F) = \sum_{L \in \Theta} L$, where Θ is the collection of subfunctors of F of finite length and with composition factors the simple objects of the form S_M , with M a non projective indecomposable module.

Definition 3.14. *Let F be a finitely presented functor. Then F is torsion free if and only if $t_H(F) = 0$.*

We end the paper with the following result, whose proof is essentially in [19, Lemma 5.4.]

Lemma 3.15. *Let F be a functor in $\text{mod}(\text{mod}\Lambda)$. Then F has projective dimension less or equal to one, if and only if, F is torsion free.*

ACKNOWLEDGEMENTS. The second author thanks CONACYT for giving him financial support during his graduate studies.

This paper is in final form and no version of it will be submitted for publication elsewhere.

REFERENCES

- [1] M. Auslander, Representation Dimension of Artin Algebras I, Queen Mary College Mathematics Notes, 1971., *Selected Works of Maurice Auslander, Part 1 AMS*, 505 - 574.
- [2] M. Auslander, Representation Theory of Artin Algebras I, *Communications In Algebra*, **1(3)** (1974), 177 - 268.

- [3] M. Auslander and Ragnar-Olaf Buchweitz, The Homological Theory of Maximal Cohen-Macaulay Approximations, *Société Mathématique de France Mémoire*, **38** (1989), 5 - 37.
- [4] M. Auslander and I. Reiten, Applications of Contravariantly Finite Subcategories, *Advances in Mathematics*, **86**(1) (1991), 111 - 151.
- [5] M. Auslander and I. Reiten, Stable equivalence of Artin algebras, Proceedings of the Conference on Orders, Group Rings and Related Topics (Ohio State Univ., Columbus, Ohio, 1972), *Lecture Notes in Math. Springer, Berlin*, **353** (1973), 8 - 71.
- [6] M. Auslander and I. Reiten, Stable Equivalence of Dualizing R -Varieties, *Advances in Mathematics*, **12**(3) (1974), 306 - 366.
- [7] M. Auslander, I. Reiten and S. O. Smalø, Representation Theory of Artin Algebras, *Cambridge University Press*, **36** (1995).
- [8] M. Auslander and Ø. Solberg, Relative homology and representation theory I. Relative cotilting theory, *Communications in Algebra*, **21**(9) (1993), 3033 - 3097.
- [9] M. Auslander and Ø. Solberg, Relative homology and representation theory I. Relative homology and homologically finite subcategories, *Communications in Algebra*, **21**(9) (1993), 2995 - 3031.
- [10] K. Bongartz, Tilted algebras, M. Auslander (ed.) E. Lluís (ed.), Representations of Algebras. Proc. ICRA III, *Lecture Notes in Mathematics, Springer*, **903** (1981), 26 - 38.
- [11] S. Brenner and M. C. R. Butler, Generalisations of the Bernstein-Gelfand-Ponomarev reflection functors, in Proc. ICRA II (Ottawa, 1979), *Lecture Notes in Math. Springer-Verlag, Berlin, Heidelberg, New York*, **832** (1980), 103 - 169.
- [12] E. Cline, B. Parshall and L. Scott, Derived categories and Morita theory, *J. Algebra*, **104** (1986), 397 - 409.
- [13] R. Fossum, H. B. Foxby, P. Griffith and I. Reiten, Minimal injective resolutions with applications to dualizing modules and Gorenstein modules, *Inst. Hautes Études Sci. Publ. Math.*, **45** (1975), 193 - 215.
- [14] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, *London Math. Soc. Lecture Note Soc.*, **119** (1988).
- [15] R. Martínez-Villa, Applications of Koszul algebras: The preprojective algebra, *Representation Theory of Algebras, CMS Conf. Proc. Amer. Math. Soc., Providence, RI*, **18** (1996), 487 - 504.
- [16] R. Martínez-Villa and G. Montaña-Bermúdez, Triangular matrix and Koszul algebras, *Int. J. Algebra*, **1**, **9 - 12** (2007), 441 - 467.
- [17] R. Martínez-Villa and M. Ortiz, Tilting Theory and Functor Categories I. Classical Tilting, *preprint, arXiv:math.RT/0165593*.
- [18] R. Martínez-Villa and M. Ortiz, Tilting Theory and Functor Categories II. Generalized Tilting, *preprint, arXiv:math.RT/0165605*.
- [19] R. Martínez-Villa and Ø. Solberg, Graded and Koszul categories, *Appl. Categ. Structures*, doi:10.1007/s10485-009-9191-6, *in press*.
- [20] R. Martínez-Villa and Ø. Solberg, Artin and Schelter regular algebras and categories, *Journal of Pure and Applied Algebra*, *in press*.
- [21] R. Martínez-Villa and Ø. Solberg, Noetherianity and Gelfand-Kirillov dimension of components, *J. Algebra*, **323** (5) (2010), 1369 - 1407.
- [22] R. Martínez-Villa and Ø. Solberg, Serre duality for Artin-Schelter regular K -categories, *Int. J. Algebra*, **3**(5-9) (2009), 355 - 375.
- [23] J. Miyachi, Derived Categories With Applications to Representations of Algebras, www.u-gakugei.ac.jp/~miyachi/papers/ChibaSemi.pdf.

INSTITUTO DE MATEMÁTICAS UNAM, UNIDAD MORELIA, MEXICO
E-mail address: `mvilla@matmor.unam.mx`

INSTITUTO DE MATEMÁTICAS UNAM, UNIDAD MORELIA, MEXICO
E-mail address: `mortiz@matmor.unam.mx`
URL: <http://www.matmor.unam.mx>